BILINEARIZATION OF $N=1$ SUPERSYMMETRIC MODIFIED KDV EQUATIONS

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Abstract

Two different types of $N=1$ modified KdV equations are shown to possess N soliton solutions. The soliton solutions of these equations are obtained by casting the equations in the bilinear forms using the supersymmetric extension of the Hirota method. The distinguishing features of the soliton solutions of $N=1$ mKdV and $N=1$ mKdV B equations are discussed.

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1. Introduction

The study of supersymmetric integrable systems has assumed great importance in recent years. This has been motivated by a number of factors, one of them being the possibility that a fundamental physical theory must be supersymmetric. In the context of integrable models, this has resulted in the supersymmetrization of a number of integrable equations and the extension of the methodologies involved in the study of integrable hierarchies to the supersymmetric framework. Investigations have also been made in recent times to obtain τ functions for supersymmetric integrable hierarchies as they are essential in the context of integrability as well as in generating soliton solutions [1, 2, 3, 4]. Integrability of a nonlinear differential equation implies the existence of nondissipating wave like solutions called solitons. Of the various known methods to generate soliton solutions, the Hirota bilinear method is the most direct and elegant one [5]. τ functions play the central role in obtaining soliton solutions in the Hirota formalism. Another significance of the τ function is its role as the effective action in quantum field theories.

Extension of the bilinear formalism to supersymmetric systems has been a comparatively recent development. A number of supersymmetric systems in $N = 1$ superspace have been bilinearized [2, 3] and their soliton solutions constructed via this technique. For $N = 2$ integrable systems, in a recent work, the N soliton solutions of the supersymmetric KdV of Inami and Kanno [6] have found through the super Hirota formalism [4]. However, several $N = 1$ equations still remain which have not been bilinearized to obtain N soliton solutions. $N = 1$ modified KdV and modified KP hierarchies are such important examples, related to the supersymmetric KdV and KP hierarchies.

In this paper we consider $N = 1$ mKdV equation [7] involving spin 1/2 super field $\psi(x, \theta, t)$,

$$\partial_t\psi + D^6\psi - 3\psi D^3\psi D\psi - 3(D\psi)^2 D^2\psi = 0 \tag{1.1}$$

and $N = 1$ mKdV B equation having spin 1 super field $\Psi(x, \theta, t)$

$$\partial_t \Psi + D^6 \Psi - 2D^2 \Psi^3 = 0 \tag{1.2}$$

and show that these equations can be bilinearised following the Hirota method and possess N soliton solutions. In (1.1,1.2) D denotes the superderivative defined by

$$D = \frac{\partial}{\partial \theta} + \theta \frac{\partial}{\partial x} \tag{1.3}$$

While the bilinear forms of the first system (1.1) involve the super Hirota operator [2], it will be seen that the for the second system (1.2) will involve only the bosonic or ordinary Hirota operator in its bilinear equations. But the importance of $N = 1$ mKdV B vis a vis $N = 1$ KdV B arises because of their connections with superstring theory from the point of view of matrix models [8]. The consequence of the presence or absence of the super Hirota operator in the bilinear forms of these two systems will be reflected in the soliton solutions of the equations in the role played by the fermionic parameters and it will also be shown that they possess distinctly different soliton solutions. These two equations, in fact, have different origins as the reductions of $N = 2$ systems.

The organization of the paper is as follows. In sections 1 and 2, the bilinearization of the $N = 1$ KdV and $N = 1$ mKdV B equations respectively will be demonstrated. Section 3 will be a discussion of the one soliton solution for both equations. In section 4, the existence of two and higher solitons will be shown. Section 5 is the concluding one.

2. $N = 1$ mKdV equation

The $N = 1$ mKdV equation (1.1) is related to the $N = 1$ KdV equation of Manin-Radul-Mathieu [9, 10]

$$\partial_t \phi + D^6 \phi + 3D^2(\phi D \phi) = 0 \tag{2.1}$$

through the super Miura transformation [11]

$$\phi = D^2\psi + \psi D\psi \tag{2.2}$$

Here the superfields ψ and ϕ are both fermionic having spins 3/2 and 1/2 respectively. The $N = 1$ mKdV also follows as an $N = 1$ reduction of the $N = 2$ mKdV system [6], which has the explicit form

$$\partial_t\psi_1 = -D\left[D^5\psi_1 + 3\psi_1 D^2\psi_2 D\psi_2 - \frac{1}{2}(D\psi_1)^3 - \frac{3}{2}D\psi_1(D\psi_2)^2\right] \tag{2.3}$$

$$\partial_t\psi_2 = -D\left[D^5\psi_2 + 3\psi_2 D^2\psi_1 D\psi_1 - \frac{1}{2}(D\psi_2)^3 - \frac{3}{2}D\psi_2(D\psi_1)^2\right] \tag{2.4}$$

By imposing the constraint $\psi_1 = -\psi_2 = \psi$, (2.3,2.4) immediately reduce to (1.1). But we will see that (1.2) is not a direct reduction of $N = 2$ mKdV. It has a different origin, rather follows from $N = 2$ KdV of the Inami Kanno type.

In order to cast the $N = 1$ mKdV equation in the bilinear form the superfield ψ is written in terms of τ functions as

$$\psi = D\log\frac{\tau_1}{\tau_2} \tag{2.5}$$

In terms of the supersymmetric Hirota derivative $\mathbf{S}$ [2], defined by

$$\mathbf{SD}_x^n f.g = (D_{\theta_1} - D_{\theta_2})(\partial_{x_1} - \partial_{x_2})^n f(x_1,\theta_1)g(x_2,\theta_2)|_{\substack{x_1 = x_2 = x \\ \theta_1 = \theta_2 = \theta}} \tag{2.6}$$

where $\mathbf{D}$ is ordinary (or bosonic) Hirota derivative, the $N = 1$ mKdV equations can be cast in the following bilinear forms:

$$(\mathbf{SD}_t + \mathbf{SD}_x^3)(\tau_1.\tau_1) = 0 \tag{2.7}$$

$$(\mathbf{SD}_t + \mathbf{SD}_x^3)(\tau_2.\tau_2) = 0 \tag{2.8}$$

$$\mathbf{D}_x^2(\tau_1.\tau_2) = 0 \tag{2.9}$$

$$\mathbf{SD}_x(\tau_1.\tau_2) = 0 \tag{2.10}$$

In obtaining the bilinear equations above, use has been made of the identities

$$D^4 \log(\tau_1\tau_2) + \left(D^2 \log \frac{\tau_1}{\tau_2}\right)^2 = \frac{\mathbf{D}^2(\tau_1.\tau_2)}{\tau_1\tau_2} \tag{2.11}$$

and

$$D^3 \log(\tau_1\tau_2) + D^2 \log \frac{\tau_1}{\tau_2} D \log \frac{\tau_1}{\tau_2} = \frac{\mathbf{SD}(\tau_1.\tau_2)}{\tau_1\tau_2} \tag{2.12}$$

The bilinear forms (2.7,2.8,2.9,2.10) involve supersymmetric operator $\mathbf{S}$ in addition to the bosonic Hirota derivative $\mathbf{D}$. The consequence of the presence of $\mathbf{S}$ operator will be apparent in the higher soliton solutions through the nontrivial relations between bosonic and fermionic parameters.

3. $N = 1$ mKdV B equation

Just as the $N = 1$ mKdV equation (1.1) is related to the $N = 1$ KdV equation of Manin-Radul-Mathieu through super Miura transformation (2.2), the $N = 1$ mKdV B equation (1.2) is related to the $N = 1$ KdV B equation [8, 12]

$$\partial_t \Phi + D^6\Phi + 6D^2\Phi D\Phi = 0 \tag{3.1}$$

$\Phi(x, \theta)$ being a spin 3/2 superfield, through a Miura tranformation,

$$D\Phi = D^2\Psi - \Psi^2 \tag{3.2}$$

Note that the Miura transformation (3.2) is nonlocal. Interestingly, in the bosonic limit (1.2) also reduces to the modified KdV equation and is invariant under the supersymmetric transformation

$$\delta\Psi^b = \eta\Psi^f \;\; ; \;\; \delta\Psi^f = \eta\partial_x\Psi^b \tag{3.3}$$

As mentioned earlier, the equation (1.2) directly results from $N = 2$ KdV equations [6]

$$\partial_t U + D^6 U + 3D^2((DU)V) - \frac{1}{2}D^2(U^3) = 0 \tag{3.4}$$

$$\partial_t V + D^6 V - 3D^2(V(DV)) - \frac{3}{2}D^2(VU^2) + 3D^2(VD^2U) = 0 \tag{3.5}$$

where U and V are superfields of conformal spin 1 and 3/2 respectively. It is straightforward to show that in the limit $V = 0$ reduces to the $N = 1$ mKdV B equation by identifying $U = \Psi$. Interestingly, in yet another limit, namely $V = DU$ both the equations (3.4,3.5) acquire the same form which is identical to the $N = 1$ mKdV B equation.

With the substitution of

$$\Psi = D^2 \log \frac{\tau_1}{\tau_2} \tag{3.6}$$

the $N = 1$ mKdV B can be cast in the bilinear equations

$$(\mathbf{D}_x\mathbf{D}_t + \mathbf{D}_x^4)(\tau_1.\tau_1) = 0 \tag{3.7}$$

$$(\mathbf{D}_x\mathbf{D}_t + \mathbf{D}_x^4)(\tau_2.\tau_2) = 0 \tag{3.8}$$

$$(\mathbf{D}_x^2)(\tau_1.\tau_2) = 0 \tag{3.9}$$

The bilinearization for $N = 1$ mKdV B, however, is not unique. It possesses an alternate, but important set of bilinear forms

$$(\mathbf{D}_t + \mathbf{D}_x^3)(\tau_1.\tau_2) = 0 \tag{3.10}$$

$$\mathbf{D}_x^2(\tau_1.\tau_2) = 0 \tag{3.11}$$

The bilinear equations (3.7,3.8,3.9) while directly follow as the reduction of the bilinear forms of the $N = 1$ mKP equation

$$\partial_t U + D^6 U - \frac{1}{2}D^2(U^3) + 12\partial_y^2 D^{-2}U + 6D^2U\partial_y D^{-2}U = 0 \tag{3.12}$$

The equations (3.10,3.11) become useful to show its connection with bosonic mKdV equation. Both the equivalent bilinear equations of the $N = 1$ mKdV B equation do not involve the supersymmetric Hirota operator leading different types of solitons solutions than those of $N = 1$ mKdV equation.

4. One Soliton Solutions

The general structure of the τ-functions for the one soliton solution both the $N = 1$ mKdV equations may be written as

$$\tau_1 = 1 + \alpha e^{\eta} \tag{4.1}$$

$$\tau_2 = 1 + \beta e^{\eta} \tag{4.2}$$

α and β in (4.1,4.2) are nonzero constants and

$$\eta = kx + \omega t + \zeta\theta \tag{4.3}$$

where k and ω are the bosonic parameters and ζ is the Grassmann odd parameter. But the soliton solutions in their explicit forms will be quite different.

Substituting (4.1) and (4.2) back into the Hirota equations of the $N = 1$ mKdV equation (2.9) and (2.10), we find the non-trivial solutions provided

$$\beta = -\alpha. \tag{4.4}$$

The dispersion relation, however, follows from (2.7) and (2.8) as

$$\omega + k^3 = 0 \tag{4.5}$$

which is identical to the bosonic mKdV equation and it is found that the fermionic parameter does not play a role at the one soliton solution level. In obtaining the dispersion relation the following property of the super Hirota operator has been used.

$$\mathbf{SD}^n\left(e^{\eta_1}.e^{\eta_2}\right) = \left(k_1 - k_2\right)^n\left[-\left(\zeta_1 - \zeta_2\right) + \theta\left(k_1 - k_2\right)\right]e^{\eta_1+\eta_2} \tag{4.6}$$

The explicit form of the one soliton solution for the superfield ψ may be found by substituting the τ- functions in (2.5) and may be given as

$$\psi = \zeta\mathrm{cosech}(\phi + \gamma_0) - \theta k\mathrm{cosech}(\phi + \gamma_0) \tag{4.7}$$

where we have chosen $\phi = kx - k^3 t$ and $\alpha = -\beta = e^{\gamma_0}$, γ_0 being nonzero, real parameter. The bosonic component of ψ in (4.7) becomes indentical with the bosonic mKdV solution as expected.

For the $N = 1$ mKdV B equation (1.2), by substituting the τ functions (4.1,4.2) in the corresponding bilinear equations, lead to identical constraint between the parameters (4.4) as well as the dispersion relation (4.5). But the differerence between the soliton solutions of these two systems becomes apparent when we consider the two and higher soliton solutions.

The one soliton solution of $N = 1$ mKdV B equation in its component fields immediately follows from (3.6) as

$$\Psi = -k\text{cosech}(\phi + \gamma_0) - \theta(k\zeta)\cosh(\phi + \gamma_0)\text{cosech}^2(\phi + \gamma_0) \tag{4.8}$$

where, $\phi = kx - k^3 t$ and $\alpha = -\beta = e^{\gamma_0}$, γ_0 being nonzero, real parameter. Notice that the bosonic component in (4.8) becomes identical with that of (4.7), as expected, since both the equations reduce to the mKdV equation in the bosonic limit. But the fermion components are quite different.

5. N Soliton Solution

Existence of one soliton solution, in fact, does not ensure the exact integrability of the system. For that purpose, establishing the existence of soliton solutions upto three solitons becomes essential. The general forms of the τ functions for N soliton as before can be chosen to be same for the both the $N = 1$ mKdV equations. The τ_1 for the N soliton may be written as

$$\tau_1 = \sum_{\mu_i=0,1} exp\left(\sum_{i,j=1}^{N} \phi(i,j)\mu_i\mu_j + \sum_{i=1}^{N} \mu_i(\eta_i + \log\alpha_i)\right) \qquad (i < j) \tag{5.1}$$

where $e^{\phi(i,j)}$ and α_i are the coefficients to be determined. For convenience, we introduce $A_{ij} = e^{\phi(i,j)}$. We may write the form of the other τ function, namely τ_2 by replacing α_i by β_i and A_{ij} by B_{ij}.

In particular, for the two soliton solution τ functions acquire the form

$$\tau_1 = 1 + \alpha_1 e^{\eta_1} + \alpha_2 e^{\eta_2} + \alpha_1\alpha_2 A_{12} e^{\eta_1+\eta_2} \tag{5.2}$$

and

$$\tau_2 = 1 + \beta_1 e^{\eta_1} + \beta_2 e^{\eta_2} + \beta_1\beta_2 B_{12} e^{\eta_1+\eta_2} \tag{5.3}$$

where

$$\eta_1 = k_1 x + \omega_1 t + \zeta_1 \theta \tag{5.4}$$

and

$$\eta_2 = k_2 x + \omega_2 t + \zeta_2 \theta \tag{5.5}$$

In (5.4,5.5) the parameters k_1, k_2, ω_1 and ω_2 are bosonic, while ζ_1 and ζ_2 are fermionic ones, as before.

For the $N = 1$ mKdV equation, the bilinear equations (2.9) and (2.10) determine the following constraints on the parameters for nontrivial solution. The conditions that

$$\beta_i = -\alpha_i \tag{5.6}$$

and the dispersion relations

$$\omega_i + k_i^3 = 0 \tag{5.7}$$

for $i = 1, 2$ once again arise in the two soliton solution as in the case of one soliton. Moreover, these two bilinear equations yield the two soliton interaction terms A_{12} and B_{12} as

$$A_{12} = B_{12} = \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2} \tag{5.8}$$

In addition, (2.10) leads to a relation among the fermionic and bosonic parameters, *viz.*

$$k_1 \zeta_2 = k_2 \zeta_1 \tag{5.9}$$

The condition (5.9) is necessary in order that (2.7,2.8) are consistently satisfied. It will be seen that the constraint on the parameters (5.9) is also crucial

in demonstrating the existence of three soliton solutions for the $N = 1$ mKdV equation.

For the $N = 1$ mKdV B equation the bilinear equations (3.7,3.8,3.9) ensure the nontrivial two soliton solutions invoking the set of identical results as in $N = 1$ mKdV equation but for the last condition (5.9). In particular, the conditions on the parameter α_i and β_i (5.7), the dispersion relations (5.7) and the interaction terms A_{12} and B_{12} (5.8) become identical in both the cases. However, in contrast to the $N = 1$ mKdV, the relation among the fermionic and bosonic parameters does not arise for the $N = 1$ mKdV B equation. This fact is reflected in the structure of the bilinear equations, which are expressed only in terms of the bosonic Hirota operator $\mathbf{D}$ and will be observed for all higher soliton solutions also.

The explicit forms of τ_1 and τ_2 for the three soliton solution following (5.1) may be given as

$$\begin{aligned}\tau_1 &= 1 + \alpha_1 e^{\eta_1} + \alpha_2 e^{\eta_2} + \alpha_3 e^{\eta_3} + \alpha_1\alpha_2 A_{12} e^{\eta_1+\eta_2} + \alpha_1\alpha_3 A_{13} e^{\eta_1+\eta_3} \\ &+\alpha_2\alpha_3 A_{23} e^{\eta_2+\eta_3} + \alpha_1\alpha_2\alpha_3 A_{12}A_{13}A_{23} e^{\eta_1+\eta_2+\eta_3}\end{aligned} \tag{5.10}$$

and

$$\begin{aligned}\tau_2 &= 1 + \beta_1 e^{\eta_1} + \beta_2 e^{\eta_2} + \beta_3 e^{\eta_3} + \beta_1\beta_2 B_{12} e^{\eta_1+\eta_2} + \beta_1\beta_3 B_{13} e^{\eta_1+\eta_3} \\ &+\beta_2\beta_3 B_{23} e^{\eta_2+\eta_3} + \beta_1\beta_2\beta_3 B_{12}B_{13}B_{23} e^{\eta_1+\eta_2+\eta_3}\end{aligned} \tag{5.11}$$

where

$$\eta_i = k_{ix}x + \omega_i t + \zeta_i\theta \quad (i = 1, 2, 3) \tag{5.12}$$

Notice that (5.10,5.11) do not contain any new unknown parameter; it is expressed in terms of the parameters of the two soliton solutions only. Three soliton solutions thus verifies the consistency of the parameters determined by one and two soliton solutions. Substitution of the three soliton solutions in the bilinear equations gives rise to a set of nontrivial relations among the parameters, which determine the consistency of the solutions. In the three

soliton solution we find the conditions $\beta_i = -\alpha_i$ and the dispersion relations $\omega_i + k_i^3 = 0$ $(i = 1, 2, 3)$ for all the three solitons for both the $N = 1$ mKdV and the $N = 1$ mKdV B equations. The interaction terms for both the equations become

$$A_{ij} = B_{ij} = \frac{(k_i - k_j)^2}{(k_i + k_j)^2} \quad (i, j = 1, 2, 3; \quad i \neq j) \tag{5.13}$$

The $N = 1$ mKdV equation however admits additional constraint as in the two soliton solution on the fermionic parameters:

$$k_i \zeta_j = k_j \zeta_i \quad (i, j = 1, 2, 3; \quad i \neq j) \tag{5.14}$$

which are not found for the $N = 1$ mKdV B equation. The constraints (5.14) are essential to ensure the coefficients of the terms $e^{\eta_1+\eta_2}$, $e^{\eta_1+\eta_3}$, $e^{\eta_2+\eta_3}$, $e^{2\eta_1+\eta_2+\eta_3}$, $e^{\eta_1+2\eta_2+\eta_3}$ and $e^{\eta_1+\eta_2+2\eta_3}$ to vanish. Apart from determing the unknown parameters, three soliton solutions provide a nontrivial relation among the parameters. This identity follows from the coefficient of the term $e^{\eta_1+\eta_2+\eta_3}$ to be zero and is given by

$$\begin{aligned}
A_{12}A_{13}A_{23}[(\omega_1 + \omega_2 + \omega_3) + (k_1 + k_2 + k_3)^3][-(\zeta_1 + \zeta_2 + \zeta_3) + \theta(k_1 + k_2 + k_3)] \\
+A_{23}[(\omega_1 - \omega_2 - \omega_3) + (k_1 - k_2 - k_3)^3][-(\zeta_1 - \zeta_2 - \zeta_3) + \theta(k_1 - k_2 - k_3)] \\
+A_{13}[(\omega_2 - \omega_1 - \omega_3) + (k_2 - k_1 - k_3)^3][-(\zeta_2 - \zeta_1 - \zeta_3) + \theta(k_2 - k_1 - k_3)] \\
+A_{12}[(\omega_3 - \omega_1 - \omega_2) + (k_3 - k_1 - k_2)^3][-(\zeta_3 - \zeta_1 - \zeta_2) + \theta(k_3 - k_1 - k_2)] = 0
\end{aligned} \tag{5.15}$$

and interestingly is satisfied by using the dispersion relations and the parameters in (5.14). In fact, (5.15) makes the three soliton solution nontrivial.

Existence of N soliton solutions may be shown following the same procedure as the three soliton solutions for both the systems. Apart from verifying the unknown parameters, determined by one and two soliton solutions, N soliton solutions provide a set of identities to be satisfied involving the parameters of one and two soliton solutions. However, for odd N is found that

terms of even degree become trivial as in the three soliton solution case. By the degree of a term we mean the total number of η_i present in the exponent of a term. The most nontrivial identity for odd N follows from the coefficient of the term involving all the η_i appearing once only. For even soliton solutions, on the other hand, the terms of odd degree become trivial.

6. Conclusion

Two important dynamical systems, $N = 1$ mKdV and $N = 1$ mKdV B have been shown to possess soliton solutions. The soliton solutions of these systems have been obtained following the super extension of the Hirota bilinear formalism. It is found that the bilinear forms of $N = 1$ mKdV system involve super Hirota derivative, whereas the bilinear forms of $N = 1$ mKdV B system can be expressed in the terms of bosonic Hirota derivative only. This difference of the two systems of motion is also reflected in their soliton solutions from two soliton solutions onwards. While the former equation of motion gives rise to a set of conditions on the fermionic parameters for the nontrivial solutions to exist, the latter does not. The fermion components of the soliton solutions for these two systems become quite different. The bosonic components, however, become identical for both the cases. This is in conformation with the fact that both the systems reduce the same mKdV equation in the bosonic limit, although they have different $N = 1$ supersymmetric versions. These two equations also follow from two separate $N = 2$ dynamical systems as an $N = 1$ reduction.

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